COHOMOLOGY WITH COEFFICIENTS IN STACKS

MAMUKA JIBLADZE AND TEIMURAZ PIRASHVILI

Abstract. Cohomology of a topological space with coefficients in stacks of abelian 2-groups is invented. This theory extends the classical sheaf cohomology. An application is given to twisted sheaves.

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1. Introduction

The aim of this work is to extend classical theory of sheaf cohomology [17] to stack cohomology. Recall that sheaves usually take values in sets, but in order to define the cohomology $H^n(X,F)$, $n\in\mathbb{Z}$ for a sheaf F one needs to assume that F has values in the category of abelian groups. Of course there is an important generalization of the theory when F has values in not necessarily abelian groups, but then $H^n(X,F)$ is defined only for $n=0,1$ and perhaps $n=2$. Quite similarly, a stack $\mathcal{F}$ usually takes values in the 2-category of groupoids, which we think as 2-sets and in order to define the cohomology $H^n(X,\mathcal{F})$, $n\in\mathbb{Z}$ one needs to assume that $\mathcal{F}$ takes values in the 2-category of abelian 2-groups. Again if one restrict stacks with values in not necessarily commutative 2-groups then one can still define $H^n(X,\mathcal{F})$ but only for few values of n. The nonabelian theory will be developed elsewhere and here we restricts ourselves to the case when $\mathcal{F}$ takes values in the 2-category of abelian 2-groups.

We recall that an abelian 2-group (known also as Picard category or symmetric categorical group) is a categorification of the notion of abelian group. These objects were invented by Grothendieck and Deligne in the sixties [13]. The basic result on abelian 2-groups was proved in the thesis of Sinh [26] written under the advice of Grothendieck, and states that the 2-category of abelian 2-groups is 2-equivalent to the 2-category of 2-stage spectra, see also the recent account in [18, Appendix B]. These objects play important role in many aspects of homotopy theory, arithmetic and geometry, see for example [2], [4], [23].

In this paper for any topological space X and for any stack of 2-abelian groups we define abelian groups $H^n(X,\mathcal{F}), n\in\mathbb{Z}$. In case when $\mathcal{F}$ is a sheaf considered as a discrete stack the groups $H^*(X,\mathcal{F})$ coincide with the classical sheaf cohomology. Our cohomology shares lots of properties with sheaf cohomology, including the long exact sequence associated to a suitably defined extension of stacks and behavior on suitably defined injective objects. However unlike the classical case these properties do not characterize the groups $H^*(X,\mathcal{F})$ in a unique way. To avoid this difficulty we also introduce the abelian 2-groups

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$\mathbf{H}^n(X,\mathcal{F})$, $n\in\mathbb{Z}$. The group $H^n(X,\mathcal{F})$ can be seen as the group of connected components of $\mathbf{H}^n(X,\mathcal{F})$, $n\in\mathbb{Z}$. The 2-groups $\mathbf{H}^*(X,-)$ form a suitably defined long exact sequence and vanish on injective objects. As in the classical case, we prove that these two facts characterize stack cohomology $\mathbf{H}^n(X,-)$ in a unique way. This important result is based on existence of enough injective objects in the abelian 2-category of abelian 2-groups – a surprising fact recently discovered by the second author [24]. One of our results claims that if $\mathbb{A}$ is an abelian 2-group considered as a constant stack then the groups $H^*(X,\mathbb{A})$ are homotopy invariant. We deduce this result by proving that there is an isomorphism

$$H^*(X,\mathbb{A})\cong H^*(X,\mathsf{sp}(\mathbb{A}))$$

where on the right hand side H^* denotes the cohomology of X with coefficients in spectra as it is defined in homotopy theory [1] and $\mathsf{sp}(\mathbb{A})$ is a spectrum associated to $\mathbb{A}$ according to [18, Proposition B.12].

One needs to make the reader aware of the fact that our abelian 2-groups are not assumed to be strictly commutative as many authors do. Deligne [13] also considered stacks of abelian 2-groups, but he assumes that abelian 2-groups are strictly commutative and proves that any stack of strictly commutative 2-groups is equivalent to a chain complex of sheaves of length one and hence corresponding cohomology is isomorphic to the obvious hypercohomology groups. If we drop the strict commutativity then the classical homological algebra technique is not enough to define cohomology in this generality and we have to use the machinery of the two dimensional homological algebra, recently developed in [3], [12], [24]. In that latter paper the classical theory of the derived functors and *Ext*'s [10] was extended to the framework of abelian 2-categories [24], which are certain 2-categories having properties similar to the 2-category of abelian 2-groups. One of the our main results claims that the stack cohomology can be described using the secondary *Ext* as they defined in [24]. Of course this is a 2-categorical version of Grothendieck's result in [17].

The paper is organized as follows. After some preliminaries we introduce the Čech type cohomology with coefficients in prestacks of abelian 2-groups. We also modify Berishvili's approach to (pre)sheaf cohomology [5] for prestacks and we prove that for paracompact spaces the two approaches give equivalent theories. In the next section we prove that these cohomologies in fact depend only on associated stacks. In Section 5 we relate these objects with Ext in the 2-categorical sense. In the last section we give an application to twisted sheaves and discriminants.

In this paper we restricted ourselves to topological spaces, but this restriction is not really necessary and one can develop cohomology with coefficients in stacks of abelian 2-groups for any Grothendieck site, based on hypercovers instead of Berishvili covers. In fact in our forthcoming publication we will introduce cohomology of 2-toposes, which will generalize not only the theory developed in this paper but also topos cohomology.

2. Preliminaries on 2-dimensional algebra

2.1. **Preliminaries on abelian 2-groups and abelian 2-categories.** An abelian 2-group is a groupoid $\mathbb{A}$ equipped with a symmetric monoidal structure $+:\mathbb{A}\times\mathbb{A}\to\mathbb{A}$ such that for any object x the endofunctor $x+(-):\mathbb{A}\to\mathbb{A}$ is an equivalence of categories. The

symmetry constraints are denoted by $c_{x,y} : x + y \to y + x$. An abelian 2-group is called *strictly commutative* provided $c_{x,x} = id_x$.

It would be convenient to think of abelian 2-groups as 2-dimensional analogues of abelian groups. For any abelian 2-group $\mathbb{A}$ the set of components $\pi^0(\mathbb{A})$ (written also $\pi_0(\mathbb{A})$) of $\mathbb{A}$ has a natural abelian group structure, while the automorphism group $\pi^{-1}(\mathbb{A})$ (written also $\pi_1(\mathbb{A})$) of the zero object of $\mathbb{A}$ is commutative. Abelian 2-groups form a groupoid enriched category $\mathfrak{P}$, where 1-morphisms (called simply morphisms) are symmetric monoidal functors and 2-morphisms (called tracks) are monoidal transformations. As in any groupoid enriched category, a morphism $f : \mathbb{A} \to \mathbb{B}$ is an equivalence if and only if there is a morphism $g : \mathbb{B} \to \mathbb{A}$ and tracks $1_A \Rightarrow gf$ and $1_B \Rightarrow fg$. One easily sees that a morphism $\mathbb{A} \to \mathbb{A}_1$ of abelian 2-groups is an equivalence of abelian 2-groups if and only if $\mathbb{A} \to \mathbb{A}_1$ yields an isomorphism of abelian groups $\pi^i\mathbb{A} \to \pi^i\mathbb{A}_1$ for $i = 0, -1$.

Any abelian group considered as a discrete category is an abelian 2-group. More generally, for any homomorphism of abelian groups $f : A^{-1} \to A^0$, we have an abelian 2-group $\mathsf{K(f)}$. Objects of the category $\mathsf{K(f)}$ are just elements of A^0. If $a, b \in A^0$, then a morphism from a to b is an element $x \in A^{-1}$ such that $f(x) = b - a$. The composition and the monoidal structure in $\mathsf{K(f)}$ are induced from the addition in A^i, $i = 0, 1$. It is clear that $\pi^0\mathsf{K(f)}) = \mathsf{Coker}(f)$ and $\pi^{-1}\mathsf{K(f)}) = \mathsf{Ker}(f)$. Observe that $\mathsf{K(f)}$ is a strictly commutative 2-group. It is well-known that any strictly commutative 2-group is equivalent to an abelian 2-group $K(f)$ for some f [13].

The role of the additive group of integers in the 2-dimensional algebra is plaid by an abelian 2-group Φ. Objects of the groupoid Φ are integers; if n and m are integers then there are no morphisms between them if $m \neq n$, while the automorphism group of the object n is the cyclic group of order two $\{+1, -1\}$. The monoidal structure is induced by the group structure of integers. The associativity and unitality constraints are the identity morphisms while the commutativity constraint $n + m \to m + n$ is $(-1)^{mn}$. As we see from the definition Φ is not strictly commutative.

One easily observes that for any abelian 2-groups $\mathbb{A}$ and $\mathbb{B}$ the hom-groupoid $\mathfrak{P}(\mathbb{A}, \mathbb{B})$ has a canonical structure of an abelian 2-group [13]. The 2-category $\mathfrak{P}$ possesses kernels in the 2-dimensional sense. The following construction goes back to Gabriel and Zisman [16]. Let $f : \mathbb{A} \to \mathbb{B}$ be a morphism of 2-groups. Objects of the groupoid $\mathsf{Ker}(f)$ are pairs (a, α), where a is an object of $\mathbb{A}$ and $\alpha : 0 \to f(a)$ is a morphism in $\mathbb{B}$. A morphism $(a, \alpha) \to (b, \beta)$ in $\mathsf{Ker}(f)$ is a morphism $\gamma : a \to b$ in $\mathbb{A}$ such that $f(\gamma)\alpha = \beta$. The compositions of morphisms as well as the monoidal structure in $\mathsf{Ker}(f)$ are induced from $\mathbb{A}$. Observe that we have a canonical functor $k_f : \mathsf{Ker}(f) \to \mathbb{A}$ and a canonical track $\kappa_f : 0 \Rightarrow fk_f$, defined by

$$k_f(a, \alpha) = a, \quad \kappa_f(a, \alpha) = \alpha.$$

The following important exact sequence was first constructed by Gabriel and Zisman (see p.84 in [16])

$$0 \to \pi^{-1}(\mathsf{Ker}(f)) \to \pi^{-1}(\mathbb{A}) \to \pi^{-1}(\mathbb{B}) \to \pi^0(\mathsf{Ker}(f)) \to \pi^0(\mathbb{A}) \to \pi^0(\mathbb{B}).$$

It is functorial in f in the following sense. Let

$$\begin{tikzcd} \mathbb{B} \arrow[r, "g"] & \mathbb{B}' \\ \mathbb{A} \arrow[u, "f"] \arrow[r, "t"'] \arrow[ur, phantom, "\epsilon\Nearrow"] & \mathbb{A}' \arrow[u, "f'"'] \end{tikzcd}$$

be a diagram in the 2-category $\mathfrak{P}$. Thus $\epsilon : gf \Rightarrow f't$ is a track. Then the assignment $(a, \alpha) \mapsto (t(a), \alpha')$ defines a morphism of abelian 2-groups $\mathsf{Ker}(f) \to \mathsf{Ker}(f')$. Here α' is the following composite

$$0 \Longrightarrow g(0) \xRightarrow{g(\alpha)} gf(a) \xRightarrow{\epsilon_a} f't(a) .$$

Based on the construction of kernels of abelian 2-groups, one can introduce the notion of the kernel in any 2-category $\mathfrak{T}$ enriched in $\mathfrak{P}$ [14], [22]. Let $f : A \to B$ be a morphism in $\mathfrak{T}$. A diagram

$$\begin{tikzcd} K \arrow[r, "k"'] \arrow[rr, bend left=50, "0"] & A \arrow[r, "f"'] \arrow[u, phantom, "\kappa\Uparrow"] & B \end{tikzcd}$$

is a kernel of f if for any object $X \in \mathfrak{T}$ the induced functor

$$\xi : \mathfrak{T}(X, K) \to \mathsf{Ker}(f^X)$$

is an equivalence of abelian 2-groups. Here $f^X : \mathfrak{T}(X, A) \to \mathfrak{T}(X, B)$ is the induced morphism of abelian 2-groups and $\xi(g : X \to K) = (kg, g^*(\kappa))$. Of course these notions are compatible, meaning that for $f : \mathbb{A} \to \mathbb{B}$ in $\mathfrak{P}$, the triple $(\mathsf{Ker}(f), k_f, \kappa_f)$ is the kernel of f in $\mathfrak{P}$ in this sense. To simplify notation, we will say that K is the kernel of f. By duality one introduces cokernels. According to [29] the 2-category $\mathfrak{P}$ possesses also cokernels and is a prototype of abelian 2-categories [14], [25].

A morphism $f : A \to B$ in a 2-category $\mathfrak{T}$ is called *faithful* (resp. *cofaithful*) provided the induced functor $f^X : \mathfrak{T}(X, A) \to \mathfrak{T}(X, B)$ (resp. $f_X : \mathfrak{T}(B, X) \to \mathfrak{T}(A, X)$) is faithful. For an abelian 2-category $\mathfrak{T}$, a morphism $f : A \to B$ in $\mathfrak{T}$ is faithful (resp. cofaithful) iff the induced homomorphism of abelian groups $\pi^{-1}(\mathfrak{T}(X, A)) \to \pi^{-1}(\mathfrak{T}(X, B))$ (resp. $\pi^{-1}(\mathfrak{T}(B, X)) \to \pi^{-1}(\mathfrak{T}(A, X))$) is a monomorphism.

A morphism $f : \mathbb{A} \to \mathbb{B}$ of abelian 2-groups is faithful (resp. cofaithful) iff the induced morphism $\pi^{-1}\mathbb{A} \to \pi^{-1}\mathbb{B}$ is a monomorphism (resp. $\pi^0\mathbb{A} \to \pi^0\mathbb{B}$ is an epimorphism).

An object I of an abelian 2-category $\mathfrak{T}$ is called *injective* provided for any faithful morphism $f : A \to B$ and a morphism $g : A \to I$ there exists a morphism $h : B \to I$ and a track $hf \Rightarrow g$. Dually, an object P of an abelian 2-category $\mathfrak{T}$ is called *projective* provided for any cofaithful morphism $f : A \to B$ and a morphism $g : P \to B$ there exists a morphism $h : P \to A$ and a track $fh \Rightarrow g$. We will say that $\mathfrak{T}$ has *enough injective* (resp. *enough projective*) objects if for any object A there exists an injective (resp. projective) object I (resp. P) and a faithful (resp. cofaithful) morphism $A \to I$ (resp. $P \to A$).

It was proved in [24] that the 2-category $\mathfrak{P}$ possesses enough injective and projective objects. For example the abelian 2-group Φ is the unique (up to equivalence) small, indecomposable, projective generator of $\mathfrak{P}$ [24].

An *extension* of an object A by an object C in an abelian 2-category $\mathfrak{T}$ is a triple (i, p, α) where $i : A \to B$ and $p : B \to C$ are morphisms in $\mathfrak{T}$ and $\alpha : 0 \Rightarrow pi$ is a track, such that p is cofaithful and (A, i, α) is equivalent to the kernel of p. If this is the case, then i is faithful and C is equivalent to the cokernel of i [12], [14], [22]. We sometimes depict such a situation by

$$0 \to A \to B \to C \to 0$$

without indicating the track α.

We also need a more general notion, which is called 2-exactness [12]. For simplicity we consider only the case when $\mathfrak{T} = \mathfrak{P}$. Assume we have a diagram of abelian 2-groups and tracks

$$\begin{tikzcd} \mathbb{A} \arrow[r, "f"'] \arrow[rr, bend left, "0"] & \mathbb{B} \arrow[r, "g"'] \arrow[u, phantom, "\alpha\Uparrow"] & \mathbb{C} \end{tikzcd} .$$

We will say that it is 2-exact at $\mathbb{B}$ provided the induced morphism $\mathbb{A} \to \mathsf{Ker}(g)$ is full and essentially surjective, in other words it yields an isomorphism on π^0 and epimorphism on π^{-1}. It follows then that the sequence of abelian groups

$$\pi^i(\mathbb{A}) \to \pi^i(\mathbb{B}) \to \pi^i(\mathbb{C})$$

is exact at $\pi^i(\mathbb{B})$ for $i = 0, -1$ [12].

We recall the description of colimits of pseudofunctors following [7], pp. 192-193. For simplicity we will only consider pseudofunctors over directed partial ordered sets considered as categories, as this will be the only case we need. Let I be a directed category and let $C_{()}$ be a pseudofunctor from I to the 2-category of groupoids. It is thus given by the family of groupoids $(C_i)_{i \in I}$ indexed by objects of I, functors $\alpha_* : C_i \to C_j$ for $\alpha : i \to j$ in I, and natural isomorphisms $\Theta_{\beta,\alpha} : \beta_*\alpha_* \Rightarrow (\beta\alpha)_*$ for $\alpha : i \to j$, $\beta : j \to k$, satisfying appropriate coherence conditions. The set of objects of the colimit category $C = \mathsf{colim}_{i \in I} C_i$ is just the disjoint union of the sets of objects of C_i, $i \in I$. To describe morphisms we need some notations. For objects i_1, i_2 of the category I we let $i_1 \downarrow I \downarrow i_2$ be the category with objects pairs of arrows $i_1 \to j \leftarrow i_2$ and obvious morphisms between these. For objects $P_1 \in C_{i_1}$ and $P_2 \in C_{i_2}$, the set of morphisms from P_1 to P_2 in C is just the colimit of the functor $i_1 \downarrow I \downarrow i_2 \to \mathsf{Sets}$ which assigns to the object $(\alpha_1 : i_1 \to j, \alpha_2 : i_2 \to j)$ of $i_1 \downarrow I \downarrow i_2$ the set $Hom_{C_j}(\alpha_{1*}(P_1), \alpha_{2*}(P_2))$, and to a morphism $(\alpha_1, \alpha_2) \to (\gamma\alpha_1, \gamma\alpha_2)$ the map given by

$$\Big(\alpha_{1*}(P_1) \xrightarrow{f} \alpha_{2*}(P_2)\Big) \mapsto \Big((\gamma\alpha_1)_*(P_1) \xrightarrow{\Theta^{-1}_{\gamma,\alpha_1}} \gamma_*\alpha_{1*}(P_1) \xrightarrow{\gamma_*(f)} \gamma_*\alpha_{2*}(P_2) \xrightarrow{\Theta_{\gamma,\alpha_2}} (\gamma\alpha_2)_*(P_2)\Big).$$

Now assume that $C_{()}$ takes values in $\mathfrak{P}$, i. e. is an object of the 2-category $\mathfrak{P}^I$ of pseudofunctors from I to $\mathfrak{P}$, so that each C_i is an abelian 2-group, and the data $(\alpha_*, \Theta_{\beta,\alpha})$

are compatible with the monoidal structures on the C_i. For each pair of objects (i,j) of I we choose an object $\xi(i,j)$ and morphisms $\alpha_{i,j} : i \to \xi(i,j)$, $\beta_{i,j} : j \to \xi(i,j)$ in I. We also choose an object i_0 of I. Having such choices made we define the bifunctor

$$+ : C \times C \to C$$

as follows: on objects it is given by

$$(P_1, P_2) \mapsto (\alpha_{i,j})_*(P_1) + (\beta_{i,j})_*(P_2),$$

where P_1 and P_2 are objects of C_i an C_j respectively. This assignment has obvious extension to morphisms. The bifunctor $+$ together with the object 0_{i_0} is part of an abelian 2-group structure on C. For example, the associativity constraints are obtained by choosing, using directedness of I, objects $\xi(i,j,k)$ and morphisms $\alpha_{i,j,k} : \xi(\xi(i,j),k) \to \xi(i,j,k)$, $\beta_{i,j,k} : \xi(i,\xi(j,k)) \to \xi(i,j,k)$ for each triple (i,j,k) of objects of I. This then yields, for objects P_1, P_2, P_3 of, respectively, C_i, C_j and C_k, isomorphisms $(P_1+P_2)+P_3 \to P_1+(P_2+P_3)$ as elements of the colimit over $\xi(\xi(i,j),k) \downarrow I \downarrow \xi(i,\xi(j,k))$, using the isomorphisms Θ and the associativity constraints in $C_{\xi(i,j,k)}$.

Observe that different choices of ξ, α, β give rise to an equivalent abelian 2-group. In this way one obtains a pseudofunctor

$$\mathsf{colim} : \mathfrak{P}^I \to \mathfrak{P}.$$

One observes that

$$\pi^n(\mathsf{colim}_{i\in I} C_i) = \mathsf{colim}_{i\in I}\pi^n(C_i),\ n = 0, -1.$$

It follows that $\mathsf{colim} : \mathfrak{P}^I \to \mathfrak{P}$ is exact and for any cofinal subcategory J of I the obvious morphism $\mathsf{colim}_{j\in J} C_j \to \mathsf{colim}_{i\in I} C_i$ is an equivalence of abelian 2-groups.

2.2. **2-chain complexes.** What is important for us is that there is a way of doing homological algebra in $\mathfrak{P}$, or more generally in any abelian 2-category $\mathfrak{T}$. More precisely, a *2-cochain complex* $(\mathbb{A}_*, d, \partial)$ in $\mathfrak{P}$ is a diagram of the form

$$\begin{tikzcd}
\cdots \arrow[r] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow"] & \mathbb{A}^{n-1} \arrow[r, "d^{n-1}"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\partial^n\Uparrow"] & \mathbb{A}^n \arrow[r, "d^n"] \arrow[rr, bend right=50, "0"'] \arrow[rr, phantom, bend right=25, "\Downarrow\partial^{n+1}"] & \mathbb{A}^{n+1} \arrow[r, "d^{n+1}"] \arrow[rr, bend left=50, "0"] \arrow[rr, phantom, bend left=25, "\Uparrow"] & A^{n+2} \arrow[r] & \cdots
\end{tikzcd}$$

i. e., a sequence of abelian 2-groups $\mathbb{A}^n$, maps $d^n : \mathbb{A}^n \to \mathbb{A}^{n+1}$ and tracks $\partial^n : d^{n+1}d^n \Rightarrow 0$, $n \in \mathbb{Z}$, such that for each n the tracks

$$d^{n+1}d^nd^{n-1} \xRightarrow{d^{n+1}\partial^n} d^{n+1}0 \xRightarrow{\equiv} 0$$

and

$$d^{n+1}d^nd^{n-1} \xRightarrow{\partial^{n+1}d^{n-1}} 0d^{n-1} \xRightarrow{\equiv} 0$$

coincide.

For any 2-cochain complex $(\mathbb{A}^*, d, \partial)$ and any integer n, there is a well-defined abelian 2-group called n-th cohomology $\mathbf{H}^n(\mathbb{A}^*)$ of $\mathbb{A}^*$ (see [12]). We recall here the definition of these abelian 2-groups. Assume we have morphisms of abelian 2-groups $\mathbb{A} \xrightarrow{f} \mathbb{B} \xrightarrow{g} \mathbb{C}$ and a track $\alpha : 0 \Rightarrow gf$. Then we have the diagram

$$\begin{tikzcd} \mathbb{B} \arrow[r, "g"] & \mathbb{C} \\ \mathbb{A} \arrow[u, "f"] \arrow[r] \arrow[ur, phantom, "\alpha\Nearrow"] & 0 \arrow[u] \end{tikzcd}$$

which yields a morphism of abelian 2-groups $\alpha' : \mathsf{Ker}(f) \to \Omega\mathbb{C}$, where $\Omega\mathbb{C} = \mathsf{Ker}(0 \to \mathbb{C})$. We let $\mathsf{Ker}(f, \alpha)$ be the kernel of α' and call it the *relative kernel*. This construction makes sense in any abelian 2-category as well. In particular one can talk about relative cokernels. For a 2-cochain complex $(\mathbb{A}^*, d, \partial)$ we first take the relative kernel $\mathsf{Ker}(d^n, \partial^{n+1})$. It comes with a natural morphism $d' : \mathbb{A}^{n-1} \to \mathsf{Ker}(d^n, \partial^{n+1})$ and a track $\partial' : 0 \to d' \circ d^{n-2}$ and $\mathbf{H}^n(\mathbb{A}^*)$ is defined to be $\mathsf{Coker}(d', \partial')$ [12]. Following [3] we call $\mathbf{H}^*(\mathbb{A}^*)$ the *secondary cohomology* of $\mathbb{A}^*$.

We also put

$$H^n_U(\mathbb{A}^*) := \pi^0(\mathbf{H}^n(\mathbb{A}^*))$$

These groups are known as *Takeuchi-Ulbrich cohomology* [12] and first were defined in [28],[27]. We have an isomorphism

$$\pi^{-1}(\mathbf{H}^{n+1}(\mathbb{A}^*)) \cong H^n_U(\mathbb{A}^*)$$

and an exact sequence (called TU-exact sequence) of abelian groups

$$\cdots \to H^{n+1}(\pi^{-1}(\mathbb{A}^*)) \to H^n_U(\mathbb{A}^*) \to H^n(\pi^0(\mathbb{A}^*)) \to H^{n+2}(\pi^{-1}(\mathbb{A}^*)) \to \cdots$$

Lemma 1. *If $H^*_U(\mathbb{A}^*) = 0$, then $\mathbf{H}^*(\mathbb{A}^*)$ is equivalent to the zero object. More generally, if $f : \mathbb{A}^* \to \mathbb{B}^*$ is a morphism of 2-chain complexes such that the induced morphism $H^*_U(\mathbb{A}^*) \to H^*_U(\mathbb{B}^*)$ is an isomorphism of abelian groups then the morphism of abelian 2-groups $\mathbf{H}^*(\mathbb{A}^*) \to \mathbf{H}^*(\mathbb{B}^*)$ is an equivalence.*

Proof. A morphism of abelian 2-groups $\mathbb{C} \to \mathbb{C}'$ is an equivalence if and only if the induced morphism $\pi^i(\mathbb{C}) \to \pi^i(\mathbb{C}')$ is an isomorphism for $i = 0, -1$. Thus the result follows from the isomorphism $\pi^{-1}(\mathbf{H}^{n+1}(\mathbb{A}^*)) \cong H^n_U(\mathbb{A}^*)$. □

Of course there is also a notion of a morphism of 2-cochain complexes as well as a homotopy between two parallel morphisms of 2-cochain complexes, with expected properties. One has also an analogue of the long cohomological exact sequence in the 2-dimensional world [12]. In fact the following is one of the main results of [12]. Assume we have an extension of 2-cochain complexes

$$0 \longrightarrow \mathbb{A}^* \xrightarrow{i^*} \mathbb{B}^* \xrightarrow{p^*} \mathbb{C}^* \longrightarrow 0$$

Here we assume that for each n there are given tracks $\alpha^n : 0 \Rightarrow p^n i^n$ such that α^n and α^{n+1} are compatible in the obvious sense. Then there are morphisms $\mathbf{H}^n(\mathbb{C}^*) \to \mathbf{H}^{n+1}(\mathbb{A}^*)$, $n \in \mathbb{Z}$, and appropriate tracks such that the diagram

$$\cdots \to \mathbf{H}^n(\mathbb{A}^*) \to \mathbf{H}^n(\mathbb{B}^*) \to \mathbf{H}^n(\mathbb{C}^*) \to \mathbf{H}^{n+1}(\mathbb{A}^*) \to \cdots$$

is part of a 2-exact sequence.

2.3. **Derived 2-functors.** Let $\mathfrak{T}$ be an abelian 2-category with enough injective objects and A be an object in $\mathfrak{T}$. Following [24], an *injective resolution* of A is a morphism $A \to A^*$ of 2-cochain complexes, which induces isomorphism on secondary cohomology, where A is considered as a 2-chain complex concentrated in dimension 0 with trivial differentials $d = 0, \partial = 0$ and (A^*, d, ∂) is a 2-cochain complex with injective A^n, $n \geq 0$ and $A^n = 0$, $n < 0$. Moreover ∂^n is equal to the identity track for $n < 0$. As in the classical case, any object admits an injective resolution, which is unique up to homotopy. For any additive pseudofunctor $T : \mathfrak{T} \to \mathfrak{P}$ one obtains well-defined additive pseudofunctors $\mathsf{R}^n(T) : \mathfrak{T} \to \mathfrak{P}$, $n \in \mathbb{Z}$ (called the *secondary right derived 2-functors*) by

$$\mathbf{R}^n(T)(A) := \mathbf{H}^n(T(I^*))$$

where I^* is an injective resolution of A. If one takes the Takeuchi-Ulbrich homology instead, one gets the Takeuchi-Ulbrich right derived functors, which are denote by R^nT, $n \in \mathbb{Z}$. Then for any extension

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0, \quad \alpha : 0 \Rightarrow pi$$

the sequence

$$\cdots \to \mathbf{R}^{n-1}T(C) \to \mathbf{R}^nT(A) \to \mathbf{R}^nT(B) \to \mathbf{R}^nT(C) \to \cdots$$

is part of a 2-exact sequence of abelian 2-groups. Furthermore we have the following exact sequence of abelian groups

$$\cdots \to R^{n-1}T(C) \to R^nT(A) \to R^nT(B) \to R^nT(C) \to \cdots$$

Moreover $\mathbf{R}^nT = 0$ if $n < -1$ and $R^nT = 0$ if $n < 0$.

Proposition 2. [24] *Assume $\mathbf{T}^n : \mathfrak{T} \to \mathfrak{P}$, $n \in \mathbb{Z}$ is a system of additive pseudofunctors such that $\mathbf{T}^n = 0$, if $n < -1$. Assume the following conditions hold*

i) for any extension

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0, \quad \alpha : 0 \Rightarrow pi$$

the sequence

$$\cdots \to \mathbf{T}^n(A) \to \mathbf{T}^n(B) \to \mathbf{T}^n(C) \to \mathbf{T}^{n+1}(A) \to \cdots$$

is part of a 2-exact sequence of abelian 2-groups,

ii) for any injective I one has $\mathbf{T}^n(I) = 0$ for $n > 1$ and $\pi^0\mathbf{T}^1(I) = 0$.

Then there exists a natural equivalence of 2-functors

$$\mathbf{R}^n\mathbf{T}^0 \cong \mathbf{T}^n, \ n \in \mathbb{Z}.$$

In particular one can take the 2-functor $\mathbf{Hom}(-,B)$ and get the secondary derived 2-functors $\mathbf{Ext}^n_{\mathfrak{T}}(-,B)$ as well as the Takeuchi-Ulbrich derived functors $Ext^n_{\mathfrak{T}}(-,B)$. For $n=1$ these objects are related to extensions; for a precise result we refer the reader to [24].

3. Topological significance

In classical algebraic topology chain complexes usually arise from (pre)cosimplicial abelian groups, by taking the boundary operator to be the alternating sum of coface operators. According to [27] a similar construction works in dimension 2 as well. More precisely, a *precosimplicial object* in the 2-category $\mathfrak{P}$ is a sequence of abelian 2-groups $\mathbb{A}^n$, $n \geq 0$, morphisms of abelian 2-groups $d_i : \mathbb{A}^n \to \mathbb{A}^{n+1}$, $0 \leq i \leq n$ and tracks

$$\alpha_{i,j} : d_i \circ d_j \Rightarrow d_{j+1} \circ d_i, \quad i \leq j$$

such that for any $i \leq j \leq k$ the following diagram commutes

$$\begin{tikzcd}
 & d_{j+1}d_id_k \arrow[dr, Rightarrow, "d_{j+1}(\alpha)"] & \\
d_id_jd_k \arrow[ur, Rightarrow, "\alpha"] \arrow[d, Rightarrow, "d_i(\alpha)"'] & & d_{j+1}d_{k+1}d_i \arrow[d, Rightarrow, "\alpha"] \\
d_id_{k+1}d_j \arrow[dr, Rightarrow, "\alpha"'] & & d_{k+2}d_{j+1}d_i \\
 & d_{k+2}d_id_j \arrow[ur, Rightarrow, "d_{k+2}(\alpha)"'] &
\end{tikzcd}$$

According to Section 3 of [27] one can associate a 2-cochain complex $C^*(\mathbb{A}^*)$ to a precosimplicial object $\mathbb{A}^*$ of $\mathfrak{P}$. More precisely, we have $C^n(\mathbb{A}^*) = \mathbb{A}^n$ and $d = \sum_{i=0}^{n+1}(-)^i\partial^i$ with appropriate $\delta : d^2 \Rightarrow 0$. By abuse of notation we use the notations $\mathbf{H}^n(\mathbb{A}^*)$ and $H^n_U(\mathbb{A}^*)$ instead of $\mathbf{H}^n(C^*(\mathbb{A}^*))$ and $H^n_U(C^*(\mathbb{A}^*))$.

It is clear that any pseudofunctor from the category Δ to the 2-category $\mathfrak{P}$ gives rise to a precosimplicial object.

Recall that [1] a spectrum E is a sequence of topological spaces, or better simplicial sets, E_n and continuous maps $\Sigma E_n \to E_{n+1}$, $n \in \mathbb{Z}$. A spectrum E is an Ω-spectrum provided the induced map $E_n \to \Omega E_{n+1}$ is a weak equivalence. Any spectrum E gives rise to the (generalized) cohomology theory on topological spaces by $X \mapsto H^n(X,E)$, where X is a topological space. In case when E is an Ω-spectrum one has

$$H^n(X,E) = [X, E_n]$$

Let $k > 0$ be an integer. A spectrum E is called k-stage if $\pi_i(E) = 0$ for $i < 0$ and $i \geq k$. It is well known that if E is a 1-stage spectrum corresponding to the abelian group A (that is $\pi_0(E) = A$) then $H^*(X,E)$ coincides with the classical singular cohomology $H^*(X,A)$, which by definition is the cohomology of the cochain complex associated to the cosimplicial abelian group $A^{Sing_*(X)}$, where $Sing_*(X)$ is the singular simplicial set of X.

As we have seen for general E even the definition of $H^*(X,E)$ uses heavy machinery of homotopy theory. Unlike to the singular cohomology of X with coefficients in an abelian group A it is impossible to get these groups from the classical homological algebra means (see [9]).

The 2-dimensional algebra gives a similar result for 2-stage spectra. To state the corresponding result let us recall that by the result of [26] the 2-category $\mathfrak{P}$ is 2-equivalent to the 2-category of two-stage spectra (see also Proposition B.12 in [18]). If $\mathbb{A}$ is an abelian 2-group, we let $\mathsf{sp}(\mathbb{A})$ be the corresponding spectrum. Below is a hint how to construct $\mathsf{sp}(\mathbb{A})$.

It is well known that any abelian 2-group is equivalent to one for which the associativity and unitality constraints are identities. Let us call such abelian 2-groups strictly associative. Let $n \geq 2$ be an integer. Then the category of strictly associative abelian 2-groups and strict morphisms is equivalent to the full subcategory of the category of simplicial groups consisting of simplicial groups G_* whose Moore normalization is nontrivial only in dimensions n and $n+1$ [11]. For a strict abelian group $\mathbb{A}$ we let $T(\mathbb{A},n)$ be the corresponding simplicial group. A more direct construction of $T(\mathbb{A},n)$ can be found in [8]. So if one takes $\mathsf{sp}(\mathbb{A})_n$ to be the classifying space of $T(\mathbb{A},n)$, $n \geq 2$, one obtains the desired Ω-spectrum. Hence we have

$$\pi_n(\mathsf{sp}(\mathbb{A})) = \begin{cases} \pi^{-n}(\mathbb{A}) & n = 0,1 \\ 0, & n \neq 0,1. \end{cases}$$

Moreover, by dimension reasons $\mathsf{sp}(\mathbb{A})$ has only one nontrivial Postnikov invariant which is the homomorphism $\pi^0(\mathbb{A})/2\pi^0(\mathbb{A}) \to \pi^1(\mathbb{A})$ induced by the symmetry constraint

$$a \mapsto c_{a,a} \in Aut(a+a) \cong Aut(0) = \pi_1(\mathbb{A})$$

where the canonical isomorphism $Aut(0) \to Aut(b)$ is induced by the functor $b+(-) : \mathbb{A} \to \mathbb{A}$.

The most important spectrum is the sphere spectrum S, and the abelian group $\mathbb{Z}$ can be seen as the zeroth Postnikov truncation of S. If we take the next stage we obtain the spectrum $S_{\leq 1}$ with properties

$$\pi_0(S_{\leq 1}) = \mathbb{Z}, \quad \pi_1(S_{\leq 1}) = \mathbb{Z}/2\mathbb{Z}, \quad \pi_i(S_{\leq 1}) = 0, \ i \neq 0,1$$

From the above description of the Postnikov invariant it is clear that $\mathsf{sp}(\Phi)$ and $S_{\leq 1}$ are homotopy equivalent spectra. Observe also that if $\mathbb{A}$ is a strictly commutative abelian 2-group, then the Postnikov invariant of $\mathsf{sp}(\mathbb{A})$ is zero. Hence it splits as a product of two spectra $\mathsf{sp}(\pi^0(\mathbb{A}))$ and $\mathsf{sp}(\pi^{-1}(\mathbb{A}))[1]$ and hence we have

$$H^*(X, \mathsf{sp}(\mathbb{A}) \cong H^*(X, \pi^0(A)) \oplus H^{*+1}(X, \pi^{-1}(A))$$

For general $\mathbb{A}$ we have the following result.

Theorem 3. *Let X be a topological space and $\mathbb{A}$ be an abelian 2-group. Then one has the natural isomorphisms of abelian groups:*

$$H^*(X, \mathsf{sp}(\mathbb{A})) \cong H^*_U(\mathbb{A}^{Sing_*(X)})$$

Proof. Without loss of generality one can assume that $\mathbb{A}$ is strictly associative. Then the result follows from Theorem 4.3 of [8]. □

4. Preliminaries on presheaves and sheaves of abelian groups

4.1. **Main definitions.** Let X be a topological space. We let $OP(X)$ be the category corresponding to the partially ordered set of open subsets of X. A *presheaf* P on X is simply a contravariant functor from $OP(X)$ to the category of abelian groups $\mathbb{Ab}$. The category of presheaves on X is denoted by $\mathbb{Psh}(X)$. The category $\mathbb{Psh}(X)$ is an abelian category with enough projective and injective objects [17]. A sequence

$$0 \to P_1 \to P \to P_2 \to 0$$

is exact in $\mathbb{Psh}(X)$ if and only if for any open subset U of X the sequence

$$0 \to P_1(U) \to P(U) \to P_2(U) \to 0$$

is an exact sequence of abelian groups.

Let U be an open subset of X and let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of U. Recall that the nerve $N\mathfrak{U}$ of this cover is the simplicial space given by

$$[n] \mapsto \coprod_{i_0,\cdots,i_n\in I} U_{i_0\cdots i_n}$$

with the obvious face and degeneracy maps induced by the inclusion of open sets. Here $U_{i_0\cdots i_n} = \bigcap_{j=0}^n U_{i_j}$. If P is a presheaf on X we can apply P on $N\mathfrak{U}$ to obtain a cosimplicial abelian group $[n] \mapsto \prod_{i_0,\cdots,i_n\in I} P(U_{i_0\cdots i_n})$. The cohomology of the associated cochain complex is denoted by $H^*(\mathfrak{U}, P)$. The obvious augmentation $N\mathfrak{U} \to U$ yields the homomorphisms

$$P(U) \to H^0(\mathfrak{U}, P).$$

A presheaf P is called a *sheaf* provided for any open set U and any open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ of U the canonical homomorphism of abelian groups $P(U) \to H^0(\mathfrak{U}, P)$ is an isomorphism [17]. We let $\mathbb{Sh}(X)$ be the full subcategory of $\mathbb{Psh}(X)$ consisting of sheaves on X. It is well known that the inclusion $\mathbb{Sh}(X) \hookrightarrow \mathbb{Psh}(X)$ has a left adjoint $P \mapsto P^+$ which preserves kernels. It follows that $\mathbb{Sh}(X)$ is an abelian category with enough injective objects [17]. A sequence

$$0 \to P \to Q \to R \to 0$$

is exact in $\mathbb{Sh}(X)$ if and only if for any $x \in X$ the sequence

$$0 \to P_x \to Q_x \to R_x \to 0$$

is an exact sequence of abelian groups. Here for any presheaf F and point $x \in X$ the group F_x is defined by

$$F_x = \mathsf{colim}_{x\in U} F(U).$$

Any abelian group A gives rise to the constant presheaf with value A, which we denote by A_c. The associated sheaf A_c^+ is called *constant sheaf* and by abuse of notation will be denoted by A. According to [17] the group $\mathsf{Ext}^*_{\mathbb{Sh}(X)}(\mathbb{Z}, F)$ is called the *cohomology* of X with coefficient in a sheaf $F \in \mathbb{Sh}(X)$.

If F is a constant sheaf associated to an abelian group A and X is a polyhedron then these groups are isomorphic to the singular cohomology of X with coefficients in A [17].

Since injective objects in $\mathbb{Sh}(X)$ are quite mysterious it is helpful to use Čech cohomology. Let $\mathfrak{U}$ and $\mathfrak{V} = \{V_j\}_{j\in J}$ be two covers of X. One says that $\mathfrak{V}$ is a *refinement* of $\mathfrak{U}$ (notation $\mathfrak{V} < \mathfrak{U}$) if there is a map $\alpha : J \to I$ such that $V_j \subset U_{\alpha(j)}$ for all $j \in J$. Utilizing this map α we define a morphism of simplicial spaces

$$N\alpha : N\mathfrak{V} \to N\mathfrak{U}$$

by mapping $V_{j_0\cdots j_n}$ to $U_{\alpha(j_0)\cdots\alpha(j_n)}$. If $\beta : J \to I$ is another map with $V_j \subset U_{\beta(j)}$ for each $j \in J$, then two morphisms $N\alpha$ and $N\beta$ of simplicial spaces are homotopic. In fact the homotopy operators $h^k : N_n\mathfrak{V} \to N_{n+1}\mathfrak{U}$, $k = 0, \cdots, n$ are defined by mapping $V_{j_0\cdots j_n}$ to $U_{\alpha(j_0),\cdots,\alpha(j_k),\beta(j_k)\cdots\beta(j_n)}$. Hence for any presheaf P there are canonical homomorphisms

$$H^*(\mathfrak{U}, P) \to H^*(\mathfrak{V}, P)$$

and one can define the Čech cohomology $\check{H}^i(X, F)$ of X with coefficients in a presheaf P by

$$\check{H}^i(X, F) := \mathsf{colim}_{\mathfrak{U}} H^*(\mathfrak{U}, P)$$

where colimit is taken over all open covers. According to [17] for paracompact X and any sheaf F one has an isomorphism

$$\mathsf{Ext}^*_{\mathbb{Sh}(X)}(\mathbb{Z}, F) \cong \check{H}^i(X, F).$$

4.2. **Berishvili approach to sheaf cohomology.** For arbitrary X we still have a similar result but we have to use so-called Berishvili covers instead [5]. A *Berishvili cover* of a topological space X is a function α which assigns to some ordered tuples $(x_0, \cdots, x_n)$ of points of X an open subset $\alpha(x_0, \cdots, x_n)$ of X such that the following conditions i)-iv) hold.

i) $x_n \in \alpha(x_0, \cdots, x_n)$,

ii) if $\alpha(x_0, \cdots, x_n)$ is defined then

$$\alpha(x_0, \cdots, \hat{x}_i \cdots, x_n) := \alpha(x_0, \cdots, x_{i-1}, x_{i+1}, \cdots, x_n)$$

is also defined and $\alpha(x_0, \cdots, x_n) \subset \alpha(x_0, \cdots, \hat{x}_i \cdots, x_n)$.

iii) If $\alpha(x_0, \cdots, x_n)$ is defined and $x \in \alpha(x_0, \cdots, x_n)$, then $\alpha(x_0, \cdots, x_n, x)$ is defined too.

iv) $\alpha(x)$ is defined for any point $x \in X$.

If α and β are two Berishvili covers then we will say that α is a *refinement* of β if every time when $\alpha(x_0, \cdots, x_n)$ is defined then $\beta(x_0, \cdots, x_n)$ is also defined and $\alpha(x_0, \cdots, x_n) \subset \beta(x_0, \cdots, x_n)$.

Having a Berishvili cover α one can form the presimplicial space $B\alpha$, which is given by $[n] \mapsto \coprod \alpha(x_0, \cdots, x_n)$ where the coproduct is taken over all $(n+1)$-tuples of points $(x_0, \cdots, x_n)$ for which $\alpha(x_0, \cdots, x_n)$ is defined. Now having any presheaf P on X one defines the groups $H^*(\alpha, P)$ as the cohomology of the cochain complex $C^*(\alpha, P)$ which is

associated to the precosimplicial abelian group

$$[n] \mapsto \prod_{(x_0,\cdots,x_n)} P(\alpha(x_0,\cdots,x_n))$$

where the product is taken over all $(x_0,\cdots,x_n) \in X^{n+1}$ for which $\alpha(x_0,\cdots,x_n)$ is defined. Observe that if α is a refinement of β, then there is a canonical map of simplicial spaces $B\alpha \to B\beta$, which induces the homomorphism $C^*(\beta,P) \to C^*(\alpha,P)$ and by taking the colimit over all Berishvili covers of X one obtains the cochain complex $C^*(X,P)$, whose cohomology groups are denoted by $H^*(X,P)$. Thus

$$H^*(X,P) := \mathsf{colim}_\alpha H^*(\alpha,P).$$

Then we have the following fact.

Theorem 4. [5] *For any space X and any sheaf F there is a canonical isomorphism*

$$H^*(X,F) \cong \mathsf{Ext}^*_{\mathbb{Sh}(X)}(\mathbb{Z},F).$$

Proof. By the well-known axiomatic of derived functors the result follows from ii) and v) of Lemma 5 below. □

Recall that a presheaf P on X is elementary if there exists a collection of abelian groups P_x, $x \in X$ such that for any open set $U \in OP(X)$ one has

$$P(U) = \prod_{x \in U} P_x$$

with obvious restriction morphisms. One easily observes that P is in fact is a sheaf.

Proposition 5. [5] *i) For any Berishvili cover α and for any elementary presheaf P one has*

$$H^n(\alpha,P) = \begin{cases} P(X), & n = 0 \\ 0, & n \geq 1 \end{cases}$$

ii) If I is an injective object in $\mathbb{Sh}(X)$ then

$$H^n(X,I) = \begin{cases} I(X), & n = 0 \\ 0, & n \geq 1 \end{cases}$$

iii) If P is a presheaf such that $P^+ = 0$, then $H^*(X,P) = 0$.

iv) If P is a presheaf then $H^*(X,P) \cong H^*(X,P^+)$.

v) If $0 \to F_1 \to F \to F_2 \to 0$ is a short exact sequence of sheaves then one has a long exact sequence of abelian groups:

$$0 \to H^0(X,F_1) \to \cdots \to H^n(X,F) \to H^n(X,F_2) \to H^{n+1}(X,F_1) \to H^{n+1}(X,F) \to \cdots$$

Proof. i) Since the functors $H^n(\alpha,-) : \mathbb{Psh}(X) \to \mathbb{Ab}$ respect products, it suffices to consider the case when the collection of abelian groups $(P_x)_{x\in X}$ is nontrivial only in a given point, say at $x_0 \in X$. Then the "evaluation at x_0" gives rise to the contraction of the sequence

$$0 \to P_{x_0} \to C^0(\alpha,P) \to C^1(\alpha,P) \to \cdots$$

ii) For a sheaf F we let $\tilde{F}$ be the elementary sheaf generated by the collection of groups F_x. Then the canonical morphism of sheaves $F \to \tilde{F}$ is a monomorphism. It follows that if I is an injective object in $\mathbb{Sh}(X)$, then I is a direct summand of an elementary sheaf. Hence the result follows from i).

iii) In fact we will show that $\mathsf{colim}_\alpha C^*(\alpha, P) = 0$. Take any $f \in C^p(\alpha, P)$ and any $(x_0, \cdots, x_p) \in X^{p+1}$ for which $\alpha(x_0, \cdots, x_p)$ is defined. Since $P^+ = 0$ there exists an open neighborhood $\beta(x_0, \cdots, x_p)$ of x_p such that the image of $f(x_0, \cdots, x_p) \in P(\alpha(x_0, \cdots, x_p))$ in $P(\beta(x_0, \cdots, x_p))$ is zero. Now we extend the function β by putting

$$\beta(x_0, \cdots, x_n) = \begin{cases} \alpha(x_0, \cdots, x_n) & \text{if } n < p \\ \beta(x_0, \cdots, x_{n-1}) \bigcap \alpha(x_0, \cdots, x_n) & \text{if } n > p \text{ and } x_n \in \beta(x_0, \cdots, x_{n-1}) \end{cases}$$

One easily sees that β is a Berishvili cover which is a refinement of α. By our construction image of f in $C^p(\beta, P)$ is zero and the result follows.

iv) It is clear that the functors $H^n(\alpha, -) : \mathbb{Psh}(X) \to \mathbb{Ab}$, $n \geq 0$ form a δ-sequence of functors. Thus the same is true for $H^n(X, -) : \mathbb{Psh}(X) \to \mathbb{Ab}$, $n \geq 0$. Observe that the morphism $\xi : P \to P^+$ gives rise to the two short exact sequences of presheaves

$$0 \to P_1 \to P \to \mathsf{Im}(\xi) \to 0, \qquad 0 \to \mathsf{Im}(\xi) \to P^+ \to P_2 \to 0$$

with $P_i^+ = 0$, $i = 1, 2$. The long cohomological exact sequences together with iii) give the result.

iv) Since $F_2 = P^+$, where P fits in a short exact sequence of presheaves

$$0 \to F_1 \to F \to P \to 0,$$

the result follows from iv). $\square$

For paracompact X similar facts are true also for Čech cohomology. Hence for such X and an arbitrary $P \in \mathbb{Psh}(X)$ one has an isomorphism $\check{H}^i(X, P) \cong H^*(X, P)$, because both of them are isomorphic to $Ext_{\mathbb{Sh}(X)}(\mathbb{Z}, P^+)$. We will need more direct construction of the isomorphism $\check{H}^i(X, P) \cong H^*(X, P)$. To do this and also for later use it is convenient to use special covers and special maps of nerves in the definition of Čech cohomology. Namely we consider open covers of the form $\mathfrak{U} = (U_x)_{x \in X}$ where $x \in U_x$. If $\mathfrak{V} = (V_x)_{x \in X}$ is another such cover, we will write $\mathfrak{V} \leq \mathfrak{U}$ if $V_x \subset U_x$ for all $x \in X$. If this is the case, then we have a canonical map $N\mathfrak{V} \to N\mathfrak{U}$. Since special covers are cofinal in all covers, if we take $\mathsf{colim}_\mathfrak{U} H^*(\mathfrak{U}, P)$ where $\mathfrak{U}$ is running over special covers and the canonical maps then we obtain again the Čech cohomology $\check{H}^i(X, P)$. If $\mathfrak{U}$ is such a cover of X, then we can define a Berishvili cover α as follows. We put $\alpha(x) = U_x$, and then by induction on n, we define $\alpha(x_0, \cdots, x_n) = \alpha(x_0, \cdots, x_{n-1}) \cap \alpha(x_n)$ provided $x_n \in \alpha(x_0, \cdots, x_{n-1})$, otherwise $\alpha(x_0, \cdots, x_n)$ is not defined. One easily sees that in this way one gets $H^*(\mathfrak{U}, P) \cong H^*(\alpha, P)$. If one passes to the limit one obtains the homomorphism $\check{H}^*(X, P) \to H^*(X, P)$. As we said this map is an isomorphism if X is a paracompact space.

5. Cohomology with coefficients in prestacks

5.1. **Preliminaries on prestacks of abelian 2-groups.** We are assuming that the reader is familiar with the basic notions and results on stacks and prestacks. Everything what we need one can find in [19] or [21]. Since the terminology in these sources diverges we recall the main definitions.

A *prestack* $\mathcal{P}$ on X is a contravariant pseudofunctor from the category $OP(X)$ to the 2-category $\mathfrak{P}$. Thus it consists of the following data:

i) for each open set U an abelian 2-group $\mathcal{P}(U)$,

ii) for each pair of open sets $U \subset V$ a morphism of abelian 2-groups $r_U^V : \mathcal{P}(V) \to \mathcal{P}(U)$,

iii) for each triple of open subsets $U \subset V \subset W$ a track in $\mathfrak{P}$

$$r_U^V r_V^W \Rightarrow r_U^W$$

satisfying the well-known properties (see Definition 19.1.3 in [19]).

If $\mathcal{P}$ and $\mathcal{Q}$ are prestacks on X then a *morphism of prestacks* $f : \mathcal{P} \to \mathcal{Q}$ (*functor of prestacks* in the terminology [19]) is nothing but a pseudonatural transformation, in other words it consists of:

1) for each open set U a morphism of abelian 2-groups $f(U) : \mathcal{P}(U) \to \mathcal{Q}(U)$

2) for each pair of open sets $U \subset V$ a track in $\mathfrak{P}$:

$$f(U)r_U^V \Rightarrow r_U^V f(V)$$

satisfying the well-known properties (see Definition 19.1.4 in [19])

If $f, g : \mathcal{P} \to \mathcal{Q}$ are morphisms of prestacks, then a track $\theta : f \Rightarrow g$ (a *morphism of functors of prestacks* in the terminology of [19]) is nothing but a pseudomodification, in other words it is given by tracks $\theta(U) : f(U) \Rightarrow g(U)$ in $\mathfrak{P}$ for each open set U, satisfying the well-known condition (see Definition 19.1.5 in [19]).

Sometimes we will use the notation $a|_U$ instead of $r_U^V(a)$.

Observe that in many sources prestack is called "fibred category" (see for example [21]) while the term "prestack" is used for what we will call separated prestack.

Prestacks on X together with the natural morphisms of prestacks form an abelian 2-category $\mathfrak{PST}(X)$ [14]. Moreover $\mathfrak{PST}(X)$ possesses enough projective and injective objects. This easily follows from [24]. A morphism $\mathcal{P} \to \mathcal{Q}$ of prestacks is faithful (resp. cofaithful) if and only if for any open set U the induced morphism $\mathcal{P}(U) \to \mathcal{Q}(U)$ is a faithful (resp. cofaithful) morphism of abelian 2-groups. In particular a sequence

$$0 \to \mathcal{P}_1 \xrightarrow{i} \mathcal{P} \xrightarrow{p} \mathcal{P}_2 \to 0$$

of prestacks together with a track $\alpha : 0 \Rightarrow pi$ is an extension in $\mathfrak{PST}(X)$ if and only if for any open subset U of X the sequence

$$0 \to \mathcal{P}_1(U) \to \mathcal{P}(U) \to \mathcal{P}_2(U) \to 0$$

together with the track $\alpha(U) : 0 \Rightarrow pi(U)$ is an extension of abelian 2-groups.

Let $\mathcal{P}$ be a prestack on X, then we obtain two presheaves $\pi^0\mathcal{P}$ and $\pi^{-1}\mathcal{P}$ on X by $U \mapsto \pi^i(\mathcal{P}(U))$, $i = 0, -1$.

5.2. **Čech and Berishvili cohomology with coefficients in prestacks.** Let $\mathfrak{U} = \{U_i\}_{i\in I}$ be an open cover of an open set U. If $\mathcal{P}$ is a prestack on X one obtains a precosimplicial object in $\mathfrak{P}$:

$$[n] \mapsto \prod_{i_0,\cdots,i_n\in I} \mathcal{P}(U_{i_0\cdots i_n})$$

We let $\mathbf{H}^n(\mathfrak{U},\mathcal{P})$ be the secondary cohomology of the 2-cochain complex $C^*(\mathfrak{U},\mathcal{P})$ associated to it and we let $H^n_U(\mathfrak{U},\mathcal{P})$ be the corresponding Takeuchi-Ulbrich cohomology groups. The obvious augmentation $N\mathfrak{U}\to U$ yields the homomorphisms

$$\mathcal{P}(U)\to \mathbf{H}^0(\mathfrak{U},\mathcal{P})$$

and passing to π^i, $i=0,-1$ one obtains the homomorphisms

$$\pi^0(\mathcal{P}(U))\to H^0_U(\mathfrak{U},\mathcal{P}),\ \ \pi^{-1}(\mathcal{P}(U))\to H^{-1}_U(\mathfrak{U},\mathcal{P}).$$

Moreover, we put

$$\check{\mathbf{H}}^n(X,\mathcal{P}) := \mathsf{colim}_{\mathfrak{U}}\mathbf{H}^n(\mathfrak{U},\mathcal{P}),$$

where $\mathfrak{U}$ varies over all special covers and canonical maps between the corresponding nerves. In this way one obtains pseudofunctors $\check{\mathbf{H}}^n(X,-):\mathfrak{PST}(X)\to\mathfrak{P}$, $n\in\mathbb{Z}$. Below we use standard terminology of abelian 2-categories, see [12], [14].

Proposition 6. *i) If*

$$0\to\mathcal{P}_1\to\mathcal{P}\to\mathcal{P}_2\to 0$$

is an extension of prestacks then

$$\cdots\to\check{\mathbf{H}}^n(X,\mathcal{P}_1)\to\check{\mathbf{H}}^n(X,\mathcal{P})\to\check{\mathbf{H}}^n(X,\mathcal{P}_2)\to\check{\mathbf{H}}^{n+1}(X,\mathcal{P}_1)\to\cdots$$

is part of a 2-exact sequence of abelian 2-groups, while

$$\cdots\to\check{H}^n_U(X,\mathcal{P}_1)\to\check{H}^n_U(X,\mathcal{P})\to\check{H}^n_U(X,\mathcal{P}_2)\to\check{H}^{n+1}_U(X,\mathcal{P}_1)\to\cdots$$

is an exact sequence of abelian groups.

ii) For any prestack $\mathcal{P}$ there is an exact sequence of abelian groups

$$\cdots\to\check{H}^{n+1}(X,\pi^{-1}(\mathcal{P}))\to\check{H}^n_U(X,\mathcal{P})\to\check{H}^n(X,\pi^0(\mathcal{P}))\to\check{H}^{n+2}(X,\pi^{-1}(\mathcal{P}))\to\cdots$$

Proof. i) Let $\mathfrak{U}$ be an open cover of X. Then one has an extension of 2-chain complexes in $\mathfrak{P}$

$$0\to C^*(\mathfrak{U},\mathcal{P}_1)\to C^*(\mathfrak{U},\mathcal{P})\to C^*(\mathfrak{U},\mathcal{P}_2)\to 0.$$

Thanks to [12] we obtain the following 2-exact sequence of secondary cohomology:

$$\cdots\to\mathbf{H}^n(\mathfrak{U},\mathcal{P}_1)\to\mathbf{H}^n(\mathfrak{U},\mathcal{P})\to\mathbf{H}^n(\mathfrak{U},\mathcal{P}_2)\to\mathbf{H}^{n+1}(\mathfrak{U},\mathcal{P}_1)\to\cdots$$

Since the filtered colimit of 2-exact sequences of abelian 2-groups remains 2-exact the result follows. A similar argument based on the TU-exact sequence for the 2-chain complex $C^*(\mathfrak{U},\mathcal{P})$ gives ii).

∎

Observe that presheaves can be considered as discrete prestacks. So we have an obvious inclusion $\mathbb{Psh}(X) \subset \mathfrak{PST}(X)$ and if we restrict $\check{H}^n_U(X,-) : \mathfrak{PST}(X) \to \mathbb{Ab}$ on $\mathbb{Psh}(X)$ one obtains the usual Čech cohomology.

If one takes Berishvili covers instead we obtain the well-defined abelian 2-groups $\mathbf{H}^n(X,\mathcal{P})$ and abelian groups $H^*_U(X,\mathcal{P})$ with similar properties.

Proposition 7. *i) If*

$$0 \to \mathcal{P}_1 \to \mathcal{P} \to \mathcal{P}_2 \to 0$$

is an extension of prestacks then

$$\cdots \to \mathbf{H}^n(X,\mathcal{P}_1) \to \mathbf{H}^n(X,\mathcal{P}) \to \mathbf{H}^n(X,\mathcal{P}_2) \to \mathbf{H}^{n+1}(X,\mathcal{P}_1) \to \cdots$$

is part of a 2-exact sequence of abelian 2-groups, while

$$\cdots \to H^n_U(X,\mathcal{P}_1) \to H^n_U(X,\mathcal{P}) \to H^n_U(X,\mathcal{P}_2) \to H^{n+1}_U(X,\mathcal{P}_1) \to \cdots$$

is an exact sequence of abelian groups.

ii) For any prestack $\mathcal{P}$ there is an exact sequence of abelian groups

$$\cdots \to H^{n+1}(X,\pi^{-1}(\mathcal{P})) \to H^n_U(X,\mathcal{P}) \to H^n(X,\pi^0(\mathcal{P})) \to H^{n+2}(X,\pi^{-1}(\mathcal{P})) \to \cdots$$

iii) There is a morphism of abelian 2-groups $\check{\mathbf{H}}^*(X,\mathcal{P}) \to \mathbf{H}^*(X,\mathcal{P})$, which is an equivalence provided X is paracompact.

Proof. i) and ii) have the same proofs as in the previous case. To prove iii) observe that, there is a morphism from Čech cohomology to the Berishvili cohomology which is isomorphism for all presheaves. It follows from the 5-lemma and TU-exact sequence that it induces isomorphism

$$\check{H}^*_U(X,\mathcal{P}) \to H^*_U(X,\mathcal{P})$$

for any $\mathcal{P}$ and the result follows. □

5.3. **Cohomology with coefficients in constant and elementary prestacks.** As we said any abelian 2-group $\mathbb{A}$ gives rise to the constant prestack, denoted by $\mathbb{A}_c$.

Proposition 8. *For a polyhedron X one has an isomorphism*

$$\check{H}^*_U(X,\mathbb{A}_c) \cong H^*(X,\mathsf{sp}(\mathbb{A}))$$

Proof. We can assume that X has a triangulation T. We take $\mathfrak{U}$ to be the open cover of X formed by the open stars of vertices of T. It is well known that the nerve of this cover as a simplicial set is isomorphic to a simplicial set $s(T)$ associated to T (see for example, Section 9.9 of [15]). Thus we have isomorphisms

$$H^*(X,\mathsf{sp}(\mathbb{A})) = H^*(\mathbb{A}^{Sing_*(X)}) \cong H^*(\mathbb{A}^{s(T)}) \cong \check{H}^*(\mathfrak{U},\mathbb{A}_c).$$

This gives a natural transformation $H^*(X,\mathsf{sp}(\mathbb{A})) \to \check{H}^*(X,\mathbb{A}_c)$. Since the corresponding statement is well known for abelian groups, we can use the TU-exact sequence and 5-lemma to finish the proof. □

A prestack $\mathcal{P}$ on X is *elementary* if there exists a collection of abelian 2-groups $(\mathbb{A}_x)_{x\in X}$ such that for any open set $X \in OP(X)$ we have

$$\mathcal{P}(U) = \prod_{x\in U} \mathbb{A}_x$$

with obvious restriction morphisms.

Lemma 9. *If $\mathcal{P}$ is an elementary prestack on X, then $H^n_U(X,\mathcal{P}) = 0$ if $n > 0$. Moreover, we have $\mathbf{H}^n(X,\mathcal{P}) = 0$ for $n > 1$ and $\mathbf{H}^1(X,\mathcal{P})$ is a connected abelian 2-group.*

Proof. If $\mathcal{P}$ is an elementary prestack, then $\pi^i\mathcal{P}$ is an elementary presheaf, $i = 0,-1$. Hence $H^n(X,\pi^i\mathcal{P}) = 0$ for $n > 0$ thanks to 5. Now by the TU-exact sequence we get $H^n_U(X,\mathcal{P}) = 0$ if $n > 0$. By definition $\pi^0(\mathbf{H}^n(X,\mathcal{P})) = H^n_U(X,\mathcal{P}) = 0$ for $n > 0$. On the other hand $\pi^{-1}(\mathbf{H}^n(X,\mathcal{P})) = H^{n-1}_U(X,\mathcal{P}) = 0$ if $n > 1$ and the result follows. $\square$

6. Cohomology with coefficients in stacks

6.1. **Preliminaries on stacks.** For a prestack $\mathcal{P}$ and a point $x \in X$ we put

$$\mathcal{P}_x = \mathsf{colim}_{x\in U}\,\mathcal{P}(U).$$

Then $(-)_x : \mathfrak{PST} \to \mathfrak{P}$ is an exact pseudofunctor for any $x \in X$.

A morphism of prestacks $f : \mathcal{P} \to \mathcal{Q}$ is called a *weak equivalence* (see Definition 2.3 of [21]) if for any open subset U the induced functor $\mathcal{P}(U) \to \mathcal{Q}(U)$ is fully faithful and locally surjective on objects, in the sense that for any object $a \in \mathcal{Q}(U)$ and every $x \in U$ there exist an open set V with $x \in V \subset U$, an object $b \in \mathcal{P}(V)$ and an isomorphism $f(V)(b) \to r^U_V(a)$.

For a prestack $\mathcal{P}$ we let $\Pi^i\mathcal{P}$ be the sheaves on X associated to the presheaves $\pi^i\mathcal{P}$, $i = 0,-1$

$$\Pi^i\mathcal{P} = (\pi^i\mathcal{P})^+, \quad i = 0,-1.$$

We have

$$(\Pi^i\mathcal{P})_x \cong \pi^i(\mathcal{P}_x)$$

for all $x \in X$ and $i = 0,-1$.

Lemma 10. *If $f : \mathcal{P} \to \mathcal{Q}$ is a weak equivalence of prestacks then $\mathcal{P}_x \to \mathcal{Q}_x$ is an equivalence for all $x \in X$.*

Proof. The induced map $\pi^i\mathcal{P}(U) \to \pi^i\mathcal{Q}(U)$ is an isomorphism for $i = -1$ and a monomorphism if $n = 0$. Since

$$\pi^i(\mathcal{P}_x) = \mathsf{colim}_{x\in U}\pi^i(\mathcal{P}(U)), \quad \pi^i(\mathcal{Q}_x) = \mathsf{colim}_{x\in U}\pi^i(\mathcal{Q}(U))$$

it follows that $\mathcal{P}_x \to \mathcal{Q}_x$ yields an isomorphism $\pi^{-1}(\mathcal{P}_x) \to \pi^{-1}(\mathcal{Q}_x)$ and a monomorphism $\pi^0(\mathcal{P}_x) \to \pi^0(\mathcal{Q}_x)$. So far we used only full faithfulness of the functors $\mathcal{P}(U) \to \mathcal{Q}(U)$. Since f is locally surjective on objects the induced homomorphism $\pi^0(\mathcal{P}_x) \to \pi^0(\mathcal{Q}_x)$ is obviously epimorphism and we are done. $\square$

A prestack of abelian 2-groups $\mathcal{P}$ is called *separated* (in [21] the corresponding objects are called simply prestacks) provided for any open set U and any objects a, b of $\mathcal{P}(U)$ the presheaf $V \mapsto Hom_{\mathcal{P}(V)}(a|_V, b|_V)$ is a sheaf on U. Here $V \subset U$ varies over all open subsets. It follows that $\Pi^{-1}\,\mathcal{P} = \pi^{-1}\,\mathcal{P}$ provided $\mathcal{P}$ is separated.

A prestack of abelian 2-groups $\mathcal{P}$ is called a *stack* provided for any open set U and any open cover $\mathfrak{U} = \{U_i\}_{i\in I}$ the canonical morphism of abelian 2-groups $\mathcal{P}(U) \to \mathbf{H}^0(\mathfrak{U}, \mathcal{P})$ is an equivalence of categories. Observe that the relative kernel of the diagram obtained by taking the alternating sum of coface operators in

$$\prod_i \mathcal{P}(U_i) \rightrightarrows \prod_{i,j} \mathcal{P}(U_{ij}) \substack{\longrightarrow\\ \longrightarrow\\ \longrightarrow} \prod_{ijk} \mathcal{P}(U_{ijk})$$

is equivalent to the category od descent data [21]. Hence our definition is equivalent to the classical definition of a stack [19]. We let $\mathfrak{ST}$ be the full sub-2-category of $\mathfrak{PST}$ consisting of stacks.

It is well known that the inclusion $\mathfrak{ST} \subset \mathfrak{PST}$ has a left adjoint $\mathcal{P} \mapsto \mathcal{P}^+$ which preserves relative kernels. Hence it follows that $\mathfrak{ST}$ is an abelian 2-category with enough injective objects and the inclusion $\mathfrak{ST} \subset \mathfrak{PST}$ respects (relative) kernels [25].

Lemma 11. *i) If $\mathcal{F}$ is a stack then*

$$\Pi^{-1}\,\mathcal{F} = \pi^{-1}\,\mathcal{F}\,.$$

ii) If additionally $\pi^{-1}\,\mathcal{F} = 0$ then

$$\Pi^0\,\mathcal{F} \cong \pi^0\,\mathcal{F}\,.$$

Proof. If $\mathcal{F}$ is a stack then it is a separated prestack [21], hence $\pi^{-1}\,\mathcal{F}$ is a sheaf and $\Pi^{-1}\,\mathcal{F} = \pi^{-1}\,\mathcal{F}$. The second part is obvious. □

Lemma 12. *i) For any prestack $\mathcal{P}$ the canonical map $\mathcal{P} \to \mathcal{P}^+$ yields equivalences of abelian 2-groups*

$$\mathcal{P}_x \to (\mathcal{P}^+)_x, \quad x \in X.$$

In particular one has $\Pi^i(\mathcal{P}) = \Pi^i(\mathcal{P}^+)$.

ii) For a prestack $\mathcal{P}$ the stack $\mathcal{P}^+$ is equivalent to zero if and only if $\mathcal{P}_x$ is equivalent to zero for all $x \in X$.

iii) The pseudofunctor $(-)_x : \mathfrak{ST}(X) \to \mathfrak{P}$ is exact.

iv) If $\mathcal{P}$ and $\mathcal{Q}$ are stacks and $f : \mathcal{P} \to \mathcal{Q}$ is a morphism of stacks, then f is an equivalence if and only if $\mathcal{P}_x \to \mathcal{Q}_x$ is an equivalence for all $x \in X$.

v) A sequence of stacks $0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$ is an extension in $\mathfrak{ST}(X)$ if and only if for any $x \in X$ the sequence $0 \to \mathcal{F}_{1x} \to \mathcal{F}_x \to \mathcal{F}_{2x} \to 0$ is an extension of abelian 2-groups.

Proof. i) Recall that $\mathcal{P}^+$ is constructed in two steps [21]. First one constructs a separated prestack $\bar{\mathcal{P}}$. The objects of $\bar{\mathcal{P}}(U)$ are the same as of $\mathcal{P}(U)$, while the morphism set from $a \in \mathcal{P}(U)$ to $b \in \mathcal{P}(U)$ is the group of all sections on U of the sheaf generated by the presheaf $V \mapsto Hom_{\mathcal{P}(V)}(a|_V, b|_V)$ [21]. It follows that

$$\pi^{-1}(\bar{\mathcal{P}}) = \Pi^{-1}\,\mathcal{P}$$

and hence $\pi^{-1}(\bar{\mathcal{P}}_x) = \pi^{-1}(\mathcal{P}_x)$, for all $x \in X$. It is also clear from the description of $\bar{\mathcal{P}}$ that the map $\pi^0(\mathcal{P}(U)) \to \pi^0(\bar{\mathcal{P}}(U))$ is an epimorphism of abelian groups and an object $a \in \mathcal{P}(U)$ lies in the kernel of this homomorphism if and only if a is locally isomorphic to 0. It follows that for any $x \in X$ the induced map $\pi^0(\mathcal{P}_x) \to \pi^0(\bar{\mathcal{P}}_x)$ is an isomorphism. Hence $\mathcal{P}_x \to \bar{\mathcal{P}}_x$ is an equivalence of abelian 2-groups. Next $\bar{\mathcal{P}} \to \mathcal{P}^+$ is a weak equivalence (see Definition 2.7 [21]). Thus $\bar{\mathcal{P}}_x \to (\mathcal{P}^+)_x$ is an equivalence of abelian 2-groups and as a result $\mathcal{P}_x \to (\mathcal{P}^+)_x$ is too.

ii) It follows from the previous lemma that if $\mathcal{P}^+ = 0$ then $\mathcal{P}_x = 0$. Conversely, assume $\mathcal{P}_x = 0$ for all $x \in X$. Then $\Pi^i\,\mathcal{P} = 0 = \Pi^i(\mathcal{P}^+)$, $i = 0, 1$. So $\pi^{-1}(\mathcal{P}^+) = 0$. Hence $\Pi^0(\mathcal{P}^+) = \pi^0(\mathcal{P}^+) = 0$ and the result follows.

iii) The fact that $(-)_x$ preserves kernels is obvious, because the inclusion $\mathfrak{PST}(X) \hookrightarrow \mathfrak{ST}(X)$ does preserve kernels and $(-)_x : \mathfrak{PST}(X) \to \mathfrak{P}$ is exact. Assume $\mathcal{F}_1 \xrightarrow{i} \mathcal{F} \xrightarrow{p} \mathcal{F}_2$, together with a track $0 \Rightarrow pi$ is the cokernel of i in the abelian 2-category $\mathfrak{ST}(X)$. Then $\mathcal{F}_2 = \mathcal{P}^+$ where p is the cokernel of i in $\mathfrak{PST}(X)$. Then for an open set U the abelian 2-group $\mathcal{P}(U)$ is the cokernel of $\mathcal{F}_1(U) \to \mathcal{F}(U)$. Hence for any $x \in X$ the abelian 2-group $\mathcal{P}_x$ is the cokernel of $\mathcal{F}_{1x} \to \mathcal{F}_x$. Apply now the part i) to deduce the result.

iv) If $f : \mathcal{P} \to \mathcal{Q}$ is an equivalence then it is of course also a weak equivalence. Hence by Lemma 10 f_x is an equivalence for all $x \in X$. Conversely, assume f_x is an equivalence for all $x \in X$. Let $\mathcal{P}_1$ be the kernel of f and let $\mathcal{Q}_1$ be the cokernel of f. Then by iii) $\mathcal{P}_{1x} = \mathcal{Q}_{1x} = 0$. Hence $\mathcal{P}_1 = \mathcal{Q}_1 = 0$ by ii) and we are done.

v) By iii) the "if" part is clear. Assume $0 \to \mathcal{F}_1 \xrightarrow{i} \mathcal{F} \xrightarrow{p} \mathcal{F}_2 \to 0$, together with a track $0 \Rightarrow pi$ is a sequence of stacks such that for all $x \in X$ the sequence $0 \to \mathcal{F}_{1x} \to \mathcal{F}_x \to \mathcal{F}_{2x} \to 0$ together with induced tracks $0 \Rightarrow p_x i_x$ is an extension of abelian 2-groups. We claim that $\mathcal{F}_1 \to \mathcal{F}$ is faithful in $\mathfrak{ST}(X)$. To show this it is equivalent to show that $\mathcal{F}_1(U) \to \mathcal{F}(U)$ is faithful in $\mathfrak{P}$. Observe that we have a commutative diagram

$$\begin{tikzcd}
\pi^{-1}(\mathcal{F}_1(U)) \arrow[r] \arrow[d] & \pi^{-1}(\mathcal{F}(U)) \arrow[d] \\
\prod_{x\in U} \pi^{-1}(\mathcal{F}_{1x}) \arrow[r] & \prod_{x\in U} \pi^{-1}(\mathcal{F}_x)
\end{tikzcd}$$

Since $\pi^{-1}\,\mathcal{F}$ and $\pi^{-1}\,\mathcal{F}_1$ are sheaves the vertical arrows are monomorphisms. By assumption the bottom arrow is also a monomorphism and the claim follows. For a moment let us denote $\mathcal{F}_3$ the cokernel of $\mathcal{F}_1 \to \mathcal{F}$. Then we have a morphism $\mathcal{F}_3 \to \mathcal{F}_2$. By iii) it induces an equivalence $\mathcal{F}_{3x} \to F_{2x}$ for all $x \in X$, hence it is an equivalence by iv).

□

6.2. **Cohomology with coefficients in stacks.** In this section we show that the prestack cohomology defined in the previous section is in fact determined by associated stacks.

Proposition 13. *i) If $\mathcal{P}$ is a prestack, then*

$$\mathbf{H}^*(X, \mathcal{P}) \cong \mathbf{H}^*(X, \mathcal{P}^+)$$

ii) If

$$0 \to \mathcal{F}_1 \to \mathcal{F} \to \mathcal{F}_2 \to 0$$

is an extension of stacks then

$$\cdots \to \mathbf{H}^n(X, \mathcal{F}_1) \to \mathbf{H}^n(X, \mathcal{F}) \to \mathbf{H}^n(X, \mathcal{F}_2) \to \mathbf{H}^{n+1}(X, \mathcal{F}_1) \to \cdots$$

is part of a 2-exact sequence of abelian 2-groups, while

$$\cdots \to H^n_U(X, \mathcal{F}_1) \to H^n_U(X, \mathcal{F}) \to H^n_U(X, \mathcal{F}_2) \to H^{n+1}_U(X, \mathcal{F}_1) \to \cdots$$

is an exact sequence of abelian groups.

iii) For any stack $\mathcal{F}$ there is an exact sequence of abelian groups

$$\cdots \to H^{n+1}(X, \Pi^{-1}(\mathcal{F})) \to H^n_U(X, \mathcal{F}) \to H^n(X, \Pi^0(\mathcal{F})) \to H^{n+2}(X, \Pi^{-1}(\mathcal{F})) \to \cdots$$

iv) For paracompact X one has similar results for the groups $\check{\mathbf{H}}^*(X, \mathcal{F})$.

Proof. i) By Lemma 1 it suffices to show that $H^n_U(X, \mathcal{P}) \to H^n_U(X, \mathcal{P}^+)$ is an isomorphism for all n. To see this we use part 2 of Proposition 7. Thus we have the commutative diagram of abelian groups with exact rows

$$\begin{tikzcd}
\cdots \arrow[r] & H^{n+1}(X, \pi^{-1}(\mathcal{P})) \arrow[r] \arrow[d] & H^n_U(X, \mathcal{P}) \arrow[r] \arrow[d] & H^n(X, \pi^0(\mathcal{P})) \arrow[r] \arrow[d] & H^{n+2}(X, \pi^{-1}(\mathcal{P})) \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & H^{n+1}(X, \pi^{-1}(\mathcal{P}^+)) \arrow[r] & H^n_U(X, \mathcal{P}^+) \arrow[r] & H^n(X, \pi^0(\mathcal{P}^+)) \arrow[r] & H^{n+2}(X, \pi^{-1}(\mathcal{P}^+)) \arrow[r] & \cdots
\end{tikzcd}$$

By the 5-lemma it suffices to prove that $H^*(X, \pi^i(\mathcal{P})) \to H^*(X, \pi^i(\mathcal{P}^+))$ is an isomorphism. But this follows from part iv) of Proposition 5 together with isomorphisms $(\pi^i(\mathcal{P}))_x \cong \pi^i(\mathcal{P}_x) \cong \pi^i(\mathcal{P}^+_x) \cong (\pi^i(\mathcal{P}^+))_x$.

ii) By definition $\mathcal{F}_2 = \mathcal{P}^+$, where $\mathcal{P}$ is the cokernel of $\mathcal{F}_1 \to \mathcal{F}$ in $\mathfrak{PST}(X)$. By Proposition 7 one has 2-exact sequences involving the 2-groups $\mathbf{H}^n(X, \mathcal{F}_1)$, $\mathbf{H}^n(X, \mathcal{F})$ and $\mathbf{H}^n(X, \mathcal{P})$. Hence the result follows from part i). Similar arguments work for iii) and iv). $\square$

6.3. **Secondary Ext.** We start with the following familiar construction.

Lemma 14. *i) Fix a point $x \in X$ and an abelian 2-group $\mathbb{A}$. Define $i_x(\mathbb{A})$ to be the stack given by*

$$i_x(\mathbb{A})(U) = \begin{cases} \mathbb{A} & \text{if} \quad x \in U \\ 0, & \text{if} \quad x \notin U \end{cases}$$

Then for any stack $\mathcal{F}$ one has

$$\mathbf{Hom}_{\mathfrak{ST}(X)}(\mathcal{F}, i_x(\mathbb{A})) \simeq \mathbf{Hom}_{\mathfrak{P}}(\mathcal{F}_x, \mathbb{A})$$

ii) If $(\mathbb{A}_x)_{x \in X}$ is a collection of injective abelian 2-groups, then $\prod_{x \in X} i_x(\mathbb{A}_x)$ is an injective object in $\mathfrak{ST}(X)$.

iii) If $\mathcal{F}$ is a stack then there exists an extension

$$0 \to \mathcal{F} \to \mathcal{F}_1 \to \mathcal{F}_2 \to 0$$

with injective $\mathcal{F}_1$.

iv) Any injective stack is a direct summand of an elementary stack.

Proof. i) is the matter of a straightforward checking, ii) is a direct consequence of i) and Lemma 12. To show iii) we choose extensions of abelian 2-groups

$$0 \to \mathcal{F}_x \to \mathbb{A}_x \to \mathbb{B}_x \to 0$$

with $\mathbb{A}_x$ an injective abelian 2-group. Then by i) we have a morphism $\mathcal{F} \to \prod_{x\in X} i_x(\mathbb{A}_x)$ and we can take $\mathcal{F}_1 = \prod_{x\in X} i_x(\mathbb{A}_x)$. Finally iv) follows from the above and [6], Corollary 11.2. □

Let Φ be the Picard category constructed in [24] which we described in the introduction. For an abelian 2-group $\mathbb{A}$, the stack $(\mathbb{A}_c)^+$ is called the constant stack corresponding to $\mathbb{A}$. By abuse of notation we write $\mathbb{A}$ instead of $(\mathbb{A}_c)^+$.

Now we are in the position to prove our main theorem which relates the secondary *Ext* from 2.3 to our cohomology theory. Since $\mathfrak{ST}(X)$ has enough injective objects we can apply the construction from 2.3 to get for any stacks $\mathcal{F}_1, \mathcal{F}_2$ the abelian 2-group $\mathbf{Ext}^*_{\mathfrak{ST}(X)}(\mathcal{F}_1, \mathcal{F}_2)$. As usual with Takeuchi-Ulbrich objects we have

$$Ext^*_{\mathfrak{ST}(X)}(\mathcal{F}_1, \mathcal{F}_2) := \pi^0(\mathbf{Ext}^*_{\mathfrak{ST}(X)}(\mathcal{F}_1, \mathcal{F}_2)).$$

Theorem 15. *For any stack $\mathcal{F}$ one has a natural equivalence*

$$\mathbf{Ext}^*_{\mathfrak{ST}(X)}(\Phi, \mathcal{F}) \cong \mathbf{H}^*(X, \mathcal{F}).$$

Proof. Observe that

$$\mathbf{Hom}_{\mathfrak{ST}}(\Phi, \mathcal{F}) \cong \mathbf{Hom}_{\mathfrak{PST}}((\Phi)_c, \mathcal{F}) \cong \mathbf{Hom}_{\mathfrak{P}}(\Phi, \mathcal{F}(X)) \cong \mathcal{F}(X) \cong \mathbf{H}^0(X, \mathcal{F}).$$

By the axiomatic characterization of the secondary derived functors (see Proposition 2) it suffices to show that if $\mathcal{F}$ is an injective object then $\mathbf{H}^n(X, \mathcal{F}) = 0$ for $n > 1$ and $\mathbf{H}^1(X, \mathcal{F})$ is connected. By Lemma 14 this would follow if we prove a similar statement for elementary stacks. By ii) of Proposition 13 this reduces to the case of elementary prestacks which was handled in Lemma 9. □

Now we have all the ingredients to prove the following result.

Theorem 16. *Let $\mathbb{A}$ be an abelian 2-group. Then for any polyhedron X one has an isomorphism*

$$H^*(X, \mathsf{sp}(\mathbb{A})) \cong Ext^*_{\mathfrak{ST}(X)}(\Phi, \mathbb{A}).$$

Proof. By Theorem 15 and ii) Proposition 13 we have

$$Ext^*_{\mathfrak{ST}(X)}(\Phi, \mathbb{A}) \cong H^*_U(X, \mathbb{A}) \cong H^*_U(X, \mathbb{A}_c).$$

On the other hand $H^*(X, \mathsf{sp}(\mathbb{A})) \cong \check{H}^*_U(X, \mathbb{A}_c)$ thanks to Proposition 8. Hence the result follows from iii) Proposition 7. □

This result is a companion of the following result which is a trivial consequence of Theorem 16 [24]. Assume X is a topological space such that $\pi_i(X) = 0$ for $i \neq 1, 2$. It is well known that the fundamental groupoid of the loop space of X has a 2-group structure.

Denote this 2-group by $\mathbb{G}$ and again let $\mathbb{A}$ be the abelian 2-group corresponding to a 2-stage spectrum E. Then

$$H^*(X,E)\cong Ext^*(\Phi,\mathbb{A})$$

where Ext is now taken in the 2-category of 2-representations of $\mathbb{G}$ and the actions of $\mathbb{G}$ on Φ and $\mathbb{A}$ are trivial. Compare this result with the familiar fact that the cohomology of a $K(\Pi,1)$-space can be computed as Ext in the category of representations of the group Π.

6.4. **Line bundles, discriminant and twisted Sheaves.** One of the most important stacks of abelian 2-groups is given by line bundles on a manifold X. In this section we consider the cohomology with coefficients in this particular stack. As we will see soon this cohomology up to shift in the dimension is the same as the cohomology with coefficients in the sheaf of invertible elements, so we do not get anything new. But existence of this isomorphism depends on the fact that any invertible module over a local ring is trivial. Hence we get more interesting situation when we consider more general objects than manifolds. Let $(X,\mathcal{O}_X)$ be a ringed space. Thus $\mathcal{O}_X$ is a sheaf of commutative rings on X. Then we have a stack $\mathcal{L}$ of invertible $\mathcal{O}_X$-modules. This stack assigns to an open set U of X the groupoid of invertible $\mathcal{O}_X(U)$-modules and their isomorphisms. The tensor product equips this stack with a structure of a stack of abelian 2-groups. So we have well-defined groups $H^*_U(X,\mathcal{L}_X)$. It is clear that we have a canonical isomorphism of sheaves

$$\Pi^{-1}(\mathcal{L}_X)\cong {\mathcal{O}^*}_X$$

and for any $x\in X$ one has an isomorphism of abelian groups

$$(\Pi^0(\mathcal{L}_X))_x\cong \mathsf{Pic}(\mathcal{O}_x).$$

The TU-exact sequence gives us the following exact sequences of abelian groups

$$\cdots\to H^{n+1}(X,{\mathcal{O}^*}_X)\to H^n_U(X,\mathcal{L}_X)\to H^n(X,\Pi^0(\mathcal{L}_X))\to H^{n+2}(X,{\mathcal{O}^*}_X)\to\cdots$$

In particular we have

$$H^n_U(X,\mathcal{L}_X)\cong H^{n+1}(X,{\mathcal{O}^*}_X)$$

provided $\mathcal{O}_x$ is a local ring for all x. This is so for example, when X is a scheme or a complex manifold. However for general $(X,\mathcal{O}_X)$ these groups are different. It is well known that the groups $H^*(X,{\mathcal{O}^*}_X)$ appear in many problems of geometry and hopefully the same is true for $H^n_U(X,\mathcal{L}_X)$. In this direction let us mention the following fact which is a restatement of a result of Section 19.6 [19].

Proposition 17. *The set of equivalence classes of stacks of twisted $\mathcal{O}_X$-modules is isomorphic to $H^1_U(X,\mathcal{L}_X)$.*

In fact this was proved implicitly in [19] (see Remark 19.6.4 (iii) in [19]) modulo the fact that instead of group $H^1_U(X,\mathcal{L}_X)$ they have $H^2(X,\mathcal{O}^*_X)$. However they are assuming that $\mathsf{Pic}(\mathcal{O}_x)=0$ for all x. But this last restriction they have only at the very end and the argument before that proves precisely the statement we just state.

Recall also that a *discriminant* [20] over $(X,\mathcal{O}_X)$ is a pair (L,h), where L is an invertible $\mathcal{O}_X$-module and h is a nondegenerate symmetric bilinear form $h:L\otimes L\to\mathcal{O}_X$. Here the

tensor product is taken over $\mathcal{O}_X$. A morphism from (L,h) to (L',h') is an isomorphism of $\mathcal{O}_X$-modules which is compatible with forms. The isomorphism classes of discriminants is denoted by $\mathsf{Dis}(\mathcal{O}_X)$. In this way one obtains the stack of discriminants $\mathcal{D}_X$. Since $h: L\otimes L\to \mathcal{O}_X$ is an isomorphism, we see that the stack $\mathcal{D}_X$ is nothing but the kernel of $\mathcal{L}_X \xrightarrow{2} \mathcal{L}_X$. Here 2 is used in additive notations, in multiplicative notation it is given by $L\mapsto L\otimes L$. Assume for any $x\in X$ the group $\mathsf{Pic}(\mathcal{O}_x)$ is 2-divisible, meaning that any element is of the form x^2. For instance this obviously holds if $\mathcal{O}_x$ is a local ring. Then we get an extension of stacks:

$$0\to \mathcal{D}_X\to \mathcal{L}_X \xrightarrow{2} \mathcal{L}_X\to 0$$

which yields not only a well-known short exact sequence [20]

$$0\to \mathcal{O}^*/\mathcal{O}^{*2}\to \mathsf{Dis}(X)\to {}_2\mathsf{Pic}(X)\to 0$$

but also gives a function which assigns to each invertible $\mathcal{O}_X$-module L (resp. stack T of twisted $\mathcal{O}_X$-modules) a class $d(L)\in H^1_U(X,\mathcal{D}_X)$ (resp. $d(T)\in H^2_U(X,\mathcal{D}_X)$) which vanishes if and only if there exists a "square root" of L (resp. T). Of course 2 can be replaced by any integer n. The role of the stack $\mathcal{D}$ will be play by the kernel of $\mathcal{L}_X \xrightarrow{n} \mathcal{L}_X$.

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Razmadze Mathematical Institute, Tbilisi, Georgia
E-mail address: `jib@rmi.ge`

Department of Mathematics, University of Leicester, University Road, Leicester, LE1 7RH, UK
E-mail address: `tp59-at-le.ac.uk`